pdf-05 (01) Dr Satish Kumar, mathematics department.

Fourier series in complex form

Fourier series of a function $f(x)$ of period $2l$ is

$$f(x) = \frac{1}{2}a_0 + a_1\cos\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + \cdots + a_n\cos\frac{n\pi x}{l} + \cdots$$
$$+ b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + \cdots + b_n\sin\frac{n\pi x}{l} + \cdots \quad —(1)$$

We know that

$$\cos x = \frac{1}{2}[e^{ix} + e^{-ix}], \quad \sin x = \frac{1}{2}[e^{ix} - e^{-ix}](-i)$$

On putting the values of $\cos x$ and $\sin x$ in (1), we get

$$f(x) = \frac{1}{2}a_0 + \frac{a_1}{2}[e^{i\frac{\pi x}{l}} + e^{-i\frac{\pi x}{l}}] + \frac{a_2}{2}[e^{i\frac{2\pi x}{l}} + e^{-i\frac{2\pi x}{l}}] \cdots$$
$$+ \frac{a_n}{2}[e^{i\frac{n\pi x}{l}} + e^{-\frac{n\pi x}{l}}] + \frac{b_1}{2}\left[\frac{e^{\frac{i\pi x}{l}} - e^{-\frac{i\pi}{l}}}{(i)}\right] + \frac{b_2}{2}\left[\frac{e^{\frac{i.2\pi x}{l}} - e^{-\frac{2\pi x}{l}}}{i}\right]$$
$$+ \cdots + \frac{b_n}{2}\left[\frac{e^{i.\frac{n\pi x}{l}} - e^{-i.\frac{n\pi x}{l}}}{i}\right] + \cdots$$

$$f(x) = \frac{1}{2}a_0 + \left(\frac{a_1}{2} - i\frac{b_1}{2}\right)e^{\frac{i\pi x}{l}} + \left(\frac{a_2}{2} - i\frac{b_2}{2}\right)e^{\frac{2i\pi x}{l}} + \cdots \left(\frac{a_n}{2} - i\frac{b_n}{2}\right)e^{\frac{ni\pi x}{l}}$$
$$+ \left(\frac{a_1}{2} + \frac{ib_1}{2}\right)e^{-\frac{i\pi x}{l}} + \left(\frac{a_2}{2} + \frac{ib_2}{2}\right)e^{\frac{-2i\pi x}{l}} + \cdots + \left(\frac{a_n}{2} + i\frac{b_n}{2}\right)e^{-\frac{ni\pi x}{l}}$$
$$= c_0 + c_1 e^{\frac{i\pi x}{l}} + c_2 e^{\frac{2i\pi x}{l}} + \cdots + c_n e^{\frac{ni\pi x}{l}} + c_{-1}e^{-\frac{i\pi x}{l}} + c_{-2}e^{\frac{-2i\pi x}{l}}$$
$$+ \cdots + c_{-n}e^{-\frac{ni\pi x}{l}} + \cdots$$

$$f(x) = c_0 + \sum c_n e^{\frac{in\pi x}{l}} + \sum c_{-n} e^{-\frac{in\pi x}{l}}$$

where $c_0 = \frac{1}{2}a_0$, $\quad c_n = \frac{1}{2}(a_n - ib_n)$, $\quad c_{-n} = \frac{1}{2}(a_i + ib_n)$

$$c_0 = \frac{1}{2}a_0 = \frac{1}{2}\cdot\frac{1}{l}\int_0^{2l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$
$$b_n = \frac{1}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$(a_n - ib_n) = \left(\frac{1}{l}\int_0 f(x)\cos\frac{n\pi x}{l}\,dx\right) - i\left(\frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right)$$

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$$\frac{a_n - i b_n}{2} = \frac{1}{\ell}\int_0^{2\ell} f(x)\cdot \frac{\left(\cos\frac{n\pi x}{\ell} - i\sin\frac{n\pi x}{\ell}\right)}{2}\,dx$$

$$\Rightarrow \quad c_n = \frac{1}{2\ell}\int_0^{2\ell} f(x)\cdot e^{-i\cdot\frac{n\pi x}{\ell}}$$

$$c_{-n} = \frac{1}{2\ell}\int_0^{2\ell} f(x)\, e^{i\cdot\frac{n\pi x}{\ell}}\,dx.$$

Example

1. obtain the complex form of fourier series

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$$

Sol. Given $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$

$$c_0 = \frac{1}{2}\cdot\frac{1}{\ell}\int_0^{2\ell} f(x)\,dx = \frac{1}{2}\cdot\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2}\,\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_0^{\pi} f(x)\,dx$$

$$c_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot f(x)\,dx + \int_0^{\pi} 1\cdot dx\right] = \frac{1}{2}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\, e^{-inx}\,dx + \frac{1}{2\pi}\int_0^{\pi} f(x)\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} 0\cdot e^{-inx}\,dx + \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left(\frac{e^{-inx}}{-in}\right)_0^{\pi} = \frac{1}{2\pi}\cdot\frac{1}{-in}\left[e^{-in\pi} - 1\right]$$

$$= \frac{1}{2\pi n}(i)\left[\cos n\pi - i\sin n\pi - 1\right] = i\left[\frac{\cos n\pi - 1}{2n\pi}\right]$$

$c_n = -\frac{i}{2n\pi}$ if n is odd & $c_n = 0$ if n is even

$\therefore f(x) = \frac{1}{2}a_0 + c_n e^{-inx} + c_{-n} e^{inx}$, a_0, c_n, c_{-n} is just calculated above.

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Problems for Practice

Find the complex form of

① $f(x) = e^{-x}, \ -1 \le x \le 1$ ②

Ans $f(x) = \sum \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} \sinh 1 \cdot e^{in\pi x}$

② $f(x) = e^{ax}, \ -1 < x < 1$

Ans $f(x) = \frac{2}{\pi} - \frac{2}{\pi}\left[\frac{e^{2a} + e^{-2a}}{1.3} + \frac{e^{4a} + e^{-4a}}{3.5} + \cdots\right]$

③ $f(x) = \cos ax, \ -\pi < x < \pi$

Ans. $f(x) = \frac{a}{\pi} \sin a\pi + \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{iax}}{a^2 - n^2}$

Practical Harmonic analysis

Some time $f(x)$ is not given only arguments and entries have been given. In this case the process of finding the fourier series is called Harmonic analysis. Here a_0, a_n, b_n of $f(x) = \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx)$ can be calculated as

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 2 \cdot \frac{1}{2\pi - 0}\int_0^{2\pi} f(x)\,dx$$

$$\left[\because \text{mean} = \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$

$$\Rightarrow \boxed{a_0 = 2\left[\text{mean value of } f(x) \text{ in }]0, 2\pi[\right]}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = 2 \cdot \frac{1}{2\pi - 0}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= 2 \cdot \left[\text{mean value of } f(x)\cos nx \text{ in }]0, 2\pi[\right]$$

Similarly

$$b_n = 2\left[\text{mean value of } f(x)\sin nx \text{ in }]0, 2\pi[\right].$$

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<u>Remember</u> In Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

first harmonic $= a_1 \cos x + b_1 \sin x$

Second harmonic $= a_2 \cos 2x + b_2 \sin 2x$

$\vdots$

n^{th} harmonic $= a_n \cos nx + b_n \sin nx$

Example. Find Fourier series as far as the second harmonic to represent the function given by table below

x	0	30	60	90	120	150	180	210	240	270	300	330
$f(x)$	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

<u>Solⁿ.</u>

x°	$\sin x$	$\sin 2x$	$\cos x$	$\cos 2x$	$f(x)$	$f(x).\sin x$	$f(x)\sin 2x$	$f(x).\cos x$	$f(x).\cos 2x$
0	0	0	1	1	2.34	0	0	2.340	2.340
30	0.50	0.87	0.87	0.50	3.01	1.505	2.619	2.619	1.505
60	0.87	0.87	0.50	-0.50	3.69	3.210	3.210	1.845	-1.845
90	1.00	0	0	-1.00	4.15	4.150	0	0	-4.150
120	0.87	-0.87	-0.50	-0.50	3.69	3.210	-3.210	-1.845	-1.845
150	0.50	-0.87	-0.87	0.50	2.20	1.100	-1.914	-1.914	1.100
180	0	0	-1	1.00	0.83	0	0	-0.830	0.830
210	-0.50	0.87	-0.87	0.50	0.51	-0.255	0.444	-0.444	0.255
240	-0.87	0.87	-0.50	-0.50	0.88	-0.766	0.766	-0.440	-0.440
270	-1.00	0	0	-1.00	1.09	-1.090	0	0	-1.090
300	-0.87	-0.87	0.50	-0.50	1.19	-1.035	-1.035	0.595	-0.595
330	-0.50	-0.87	0.87	0.50	1.64	-0.820	-1.427	1.427	0.820
					25.22	9.209	-0.547	3.353	-3.115

$a_0 = 2 \times \text{Mean of } f(x) = 2 \times \frac{25.22}{12} = 4.203$

$a_1 = 2 \times \text{Mean of } f(x)\cos x = 2 \times \frac{3.353}{12} = 0.559$

$a_2 = 2 \times \text{mean of } f(x)\cos 2x = 2 \times \frac{-3.115}{12} = -0.519$

$b_1 = 2 \times \text{Mean of } f(x)\sin x = 2 \times \frac{9.209}{12} = 1.535$

$b_2 = 2 \times \text{Mean of } f(x)\sin 2x = 2 \times \frac{-.547}{12} = -.091$

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$b_2 = 2\times$ mean of $f(x)\sin 2x$

$= 2\times \frac{-0.547}{12} = -0.091$

Fourier series is

$f(x) = \frac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$

$= 2.1015 + 0.559\cos x - 0.519\cos 2x + \cdots$
$+ 1.535\sin x - 0.091\sin 2x + \cdots$

Dirichlet's conditions for a Fourier series.

If the function $f(x)$ for the interval $(-\pi, \pi)$

(1) is single valued (2) is bounded

(3) has at most a finite number of maxima and minima

(4) has only a finite no of discontinuous.

(5) is $f(x+2\pi) = f(x)$ for values of x outside $[-\pi, \pi]$ then

$$S_p(x) = \frac{1}{2}a_0 + \sum_{n=1}^{p} a_n\cos nx + \sum_{n=1}^{p} b_n \sin nx$$

converges to $f(x)$ as $p\to\infty$ at values of x for which $f(x)$ is continuous and to

$$\frac{1}{2}[f(x+0) + f(x-0)]$$

i.e. Suppose $f(x)$ is discontinuous at $x=c$ then

$$f(c) = \frac{1}{2}\left[\lim_{h\to 0} f(c+h) + \lim_{h\to 0} f(c-h)\right]$$

where $f(c) = \lim_{x\to c}[f(x)] = \left[\frac{a_0}{2} + \sum a_n\cos nx + \sum b_n \sin nx\right]_{x=c}$

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Advantages of Fourier Series

(1) Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist while it is not true for Taylor's series case.

(2) The fourier series is useful in expanding the periodic functions since outside the closed interval there exists a periodic extension of the function.

(3) Expansion of an oscillation function by fourier series gives all modes of oscillation, fundamental and all overtones which is extremly useful in physics.

(4) fourier series of discontinuous function is not uniformly convergent at all points.

(5) Term by term integration of a convergent fourier series is always valid and it could be valid if the series is not convergent. However term by term, differentiation of a Fourier series is not valid in most cases.

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